FP6-513663

FLUID
FLUID Image Description and analysis

Instrument: Strep
Priority: IST FET OPEN

**Deliverable 3.5**

Final report on Workpackage 3:
Representation, Tracking methodologies, and
Characterization of fluid flows from image sequences

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Due date of deliverable: 31/11/07
Actual submission date: 10/12/07

Start date of project: 01/12/04
Duration: 3 years

Organisation name of lead contractor for this deliverable: INRIA

Revision [1]

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**Project co-funded by the European Commission within the Sixth Framework Programme (2002-2006)**

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Final report on Workpackage 3:
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Characterization of fluid flows from image sequences

E. Mémin
December 3, 2007

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1 Introduction

This report aims at describing the different techniques that have been proposed within the work-package 3 of the FLUID project. The purpose of this workpackage, as defined in the document “Annex I Description of work”, was to study techniques for the tracking and characterization of fluid flow representative features. The term “tracking” is a common notion in our every day language that represents the action of following the course or trail something or someone. Even if in a traditional image analysis context this notion has been extended immediately to the issue of the visual pursuit of objects, this topic lacks from a precise definition and is employed to designate very different techniques and issues. Here the term tracking should be understood as defining processes that aim at allowing the reconstruction, within a given time range, of features trajectories from incomplete and noisy observations. It is therefore both a reconstruction process of dynamical entities and a denoising action of missing or hidden data. More precisely, in the context of this project, the goal sought was to propose methodologies allowing to extract temporal trajectories of flow features from time resolved sequences or from a set of uncorrelated images pairs.

The three tasks defined in this workpackage were

- Task 3.1: Representation and segmentation of fluid flow structures.
- Task 3.2: Tracking of salient fluid flow structures.
- Task 3.3: Characterization of turbulent structures.

The first task aimed at studying different representations of fluid structures. The word “segmentation” borrowed also to the image processing vocabulary indicates that among different representations of fluid flow velocity fields, we wanted to study basis endowing reduced compact subspaces of fluid motion description. These reduced flow representations are important in the perspective of flow control, or for tracking and experimental flow reconstruction purpose. Dense descriptors are, at the opposite, required to get access to finer details and to authorize a deeper insight. We will nevertheless see that we have been able also to propose processes for trajectories reconstruction of dense representations.
As explained earlier, the goal of the second task was to explore techniques enabling the estimation of a complete flow representation along time (i.e. representation features trajectory). We wished here to explore different techniques allowing to extract such a trajectory from a set of noisy and eventually incomplete image measurements. Two very different frameworks have been explored in that prospect.

The third task consisted to study techniques for the detection or the extraction of global coherent spatio-temporal structures of turbulent flows. This task is indeed related to the two previous tasks. One aims here at defining relevant flow descriptions and in the same time to get a description of their temporal evolution. This task will conduct us to study methods for the characterization of reduced fluid flow dynamical systems defined from image measurements.

This report will follow the plan defined by the workpackage’s tasks. In a first part we will describe the different flow representations we considered and how they can be extracted from image data. In a second section we will present the two methodological frameworks on which we rely on in this study: namely stochastic filtering methodology and variational assimilation techniques. Both frameworks have their own advantages and deficiencies. This will be briefly analysed in this section. The third part will present a technique enabling to reconstruct a reduced dynamical system describing the fluid flow evolution from image data. We will show how dynamical terms can be specified and learned from image data.

2 Fluid flow representation

During the fluid project three different kinds of flow representations have been explored in order to describe the more precisely as possible instantaneous characteristic flow features but also to authorize a tracking along time of these quantities. The chosen features must, in the one hand, endow meaningful flow descriptions and, in the other hand, permit an affordable reconstruction of their temporal evolution that is directly related to the evolution of the observed phenomenon. In that prospect, compact reduced descriptions and dense representation have been both used. The reduced representations are devoted to the description of the large persistent spatio-temporal structures of the flow (i.e. the large motion scales) whereas dense representations are dedicated to the representation of smaller motion scales.
2.1 Reduced parametric representation

This first flow representation has been designed during the PhD thesis of Anne Cuzol [12]. This representation relies on the Helmholtz decomposition which decomposes the motion field into a divergent free component and a rotational free component. These two components are respectively called the solenoidal and the irrotational components. This decomposition is defined up to boundary conditions or up to an analytic transportation (curl and divergent free) component that can be extracted using traditional motion estimation scheme. Assuming this transportation component is known and has been removed from the image sequence, the irrotational component, \( w_{irr} \), and the solenoidal component, \( w_{sol} \) are related to the divergence and the vorticity through two convolution relations:

\[
w_{irr} = K \ast \text{div} \; w_{irr} \quad \text{and} \quad w_{sol} = -K^{\perp} \ast \text{curl} \; w_{sol},
\]

where \( K \) denotes the gradient of the Green kernel associated to the Laplacian operator. The second equation of (1) is known as the Biot-Savard integral. These two equations state that the solenoidal and the irrotational components (and consequently the whole vector field) may be recovered through a convolution product knowing the divergence and the vorticity of the velocity field.

2.2 Vortex and source particles

The idea of vortex particles methods [7, 33] consists in representing the vorticity distribution of a field by a set of discrete amounts of vorticity (vortices). Using these vortices, the vorticity distribution is approximated as a weighted discrete sum:

\[
\text{curl} \; w(x) \approx \sum_i \gamma_i \delta(x - x_i),
\]

where \( x_i \) is the location of the vortex \( i \), \( \gamma_i \) the strength of the vortex and \( \delta \) the Dirac delta function. The vortices are called point vortices since they are represented by delta functions.

This discretization of the vorticity into a limited number of elements enables to evaluate the velocity field directly from the Biot-Savard integral (equ. 1). However, because of the singularity of the Green kernel gradient \( K \), the velocity field becomes unbounded if any two vortices come very close to each other. In fact, the induced field develops \( \frac{1}{r} \)-type singularities, where \( r \) is the distance to the point vortices.

These singularities can be removed by smoothing the Dirac measure with a cut-off or blob function, leading to a smoothed version of \( K \). Let \( f_\epsilon \) be such a blob function scaled by a
parameter \( \epsilon: f_\epsilon(x) = \frac{1}{\epsilon^2} f\left(\frac{x}{\epsilon}\right) \). The smoothed kernel is defined as \( K_\epsilon = K * f_\epsilon \). The amount of smoothing is determined by the value of \( \epsilon \). If \( \epsilon \to 0 \), \( f_\epsilon \) tends to the Dirac function and \( K_\epsilon \to K \).

As for the divergence map a similar source particles representation reads then:

\[
\text{div} \, w(x) \approx \sum_{i=0}^{n} \gamma_i f_{\epsilon_i}(x - x_i),
\]

where \( x_i \) denotes the center of each basis function \( f_{\epsilon_i} \), the coefficient \( \gamma_i \) is the strength associated to the particle \( i \), and \( \epsilon_i \) represents its influence domain. These parameters are free to vary from a function to another.

### 2.2.1 Estimation from consecutive images

As we saw previously, discretizing the vorticity map with vortex particles together with a Gaussian smoothing of the Dirac measure leads through Biot-Savard integral to the following representation of the solenoidal component of the motion field:

\[
w_{\text{sol}}(x) \approx \sum_{i=0}^{n} \gamma_{i,\text{sol}} K_{\epsilon_{i,\text{sol}}}^{-1}(x_{i,\text{sol}} - x),
\]

where \( K_{\epsilon_{i,\text{sol}}}^{-1} \) is a new kernel function obtained by convolving the orthogonal gradient of the Green kernel with the blob function. Obviously, a similar representation of the irrotational component can be obtained using source particles.

As a result, we exhibit an approximation of the complete motion field as weighted sums of basis functions defined by their center location and respective spatial influence.

In order to estimate a velocity expressed on this basis, this parametric motion representation is incorporated within a variation model of the luminance function. Let us note that several data model can be considered depending on the application and the type of images used to visualize the flow[3, 11, 9, 29, 28]. Assuming that such a data model holds almost everywhere on the whole image domain, \( \Omega \), leads to formalize the motion recovery as a minimization problem of the form:

\[
\mathcal{F}(I, w) = \int_{\Omega} g \left( I(x + w(x), t + 1) - h(w(x))I(x, t) \right) dx.
\]

Considering such a cost function for an unknown motion field approximated through vortex and source particles representations comes down to solve the following minimization problem:

\[
\hat{\beta} = \arg \min_{\beta} \mathcal{F}(I, w(\beta)),
\]
with \( \beta = (\{x_i^{sol}, \gamma_i^{sol}, \epsilon_i^{sol}\}_{i=1:p}, \{x_i^{irr}, \gamma_i^{irr}, \epsilon_i^{irr}\}_{i=1:q}) \). One seeks therefore the minimizer of the cost function, \( \mathcal{F} \), in terms of particles location, strength coefficients and influence domains.

Due to the peculiar form of the brightness consistency data model this minimization problem is highly non linear. To face this difficult optimization problem we have implemented a two steps estimation process. First of all, the strength coefficients and the influence domain of each particle are estimated through a multi-resolution non linear least square minimization solved with a generalized conjugated gradient known as Fletcher-Reeves method. The particles location are then moved according to a mean shift procedure [8]. This procedure allows us to move the set of vortex and sources particles toward the modes of a probability distribution related to localized image reconstruction error. Interested readers can refer to [14] where this two alternates optimization is fully described.

The figure 1 shows the results obtained for this technique for a vortex launched at the tip of an airplane wing. This laser tomographic image sequence has been obtained in a wind tunnel and shows the flow through a smoke seeding. Due to its resolution the individual particles are not visible and this image shows the evolution of a passive scalar.

![Figure 1: Vortex at tip of airplane wing (15 vortex particles) estimated from a sequence of scalar images – Sequence provided by courtesy of ONERA](image)

As shown in [14] this technique allows us to represent the large motion scales of the flow. It provides also a good instantaneous characterization of the vortical regions of the flow. An iterative estimation technique allows us to estimate the basis function coefficient but also their location. The code corresponding to different variants of this estimation can be downloaded on the FLUID web site. This corresponds to the deliverable number 3.1 entitled “Representation of fluid flow structures” and the demonstrator 3.2 "Fluid motion segmentation demonstrator". This study has been detailed in several publications [12, 13, 14, 17]. All these publications can be downloaded also on the FLUID web site. In this formulation no constraint on the temporal evolution of the basis function are embedded. As a consequence no dynamical coherence are guaranty to hold along time. Such a coherency will be impose by a dynamical tracking of these
basis functions. This tracking process is described in section 2.

2.3 Reduced non parametric representation

The second reduced parametric representation that has been used is the Proper Orthogonal Decomposition (POD) [30]. This representation is well known in fluid mechanics and routinely used in order to built a reduced representation of a flow from a sequence of motion fields. This technique is well adapted to the representation of flows with a certain degree of periodicity.

2.3.1 POD basis

POD representation has been widely used by different authors as a technique to construct approximate descriptions of the large scale or coherent structures in laminar and turbulent flows. Given an ensemble $u(x, t_i)$ obtained experimentally at $M$ different discrete instants, POD provides $M$ mutually orthogonal basis functions, or modes, $\phi_i(x)$, which are optimal with respect to average kinetic energy representation of the flux. The POD representation consists to express the velocity field as a mean flow $\bar{u}$ with fluctuations captured by a finite set of modes:

$$u(x, t) = \bar{u} + \sum_{i=1}^{M} a_i(t) \phi_i(x).$$

(7)

Considering velocity fields of finite kinetic energy, $u \in L^2$ and denoting by $(,)$ the inner product of functions defined in $L^2(S)$. The sought set of modes defines a subspace of solutions such that the projection of $u(x, t_i)$ on this subspace is optimal with respect to the average kinetic energy:

$$\frac{\langle |(u, \psi)|^2 \rangle}{\langle \psi, \psi \rangle}.$$

Here $\langle \bullet \rangle$ denotes a temporal average. The optimal basis functions $\phi$ are solution of the following eigenvalue problem:

$$\int_{S} K(x, x') \phi_k(x) dx = \phi_k(x) \lambda_k,$$

(8)

where

$$K(x, x') = \langle u(x, t) u(x', t) \rangle = \frac{1}{M} \sum_{i=1}^{M} u(x, t_i) u(x', t_i)$$

is the auto-correlation matrix built from experimental velocity fields. To solve this problem it is usually numerically easier to follow Sirovich’s Snapshots method [46] which states that each spatial mode $\phi$ can be constructed by a superposition of the velocities fields:

$$\phi_k(x) = \sum_{i=1}^{M} u(x, t_i) a_k(t_i).$$
Projecting (8) into a snapshot \( u(x, t_j) \), we obtain another eigenvalue problem, whose eigenvectors are the temporal modes \( \alpha_k \).

This technique works very well on velocity fields obtained through a Direct Numerical Simulation (DNS). For experimental velocity fields obtained by the way of PIV techniques, the presence of noise or bad experimental conditions may alter severely the results. A modification of this technique called Gappy POD enables to cope with noisy and sparse velocity measurements [24]. This scheme consists to iteratively denoise wrong or missing velocity estimates with the POD basis.

The POD representation enables a good description of the large motion scales. The spatial basis component and the temporal modal coefficients can be computed from an eigenvalue problem or from a truncated singular value decomposition [25]. Such a process requires the whole set of measurements. As opposed to the previous representation based on vortex particles, this technique is a batch technique. This representation cannot be accurately estimated from a reduced set of images. Besides, POD is adapted only for periodic flows whereas there is not such a limitation for vortex particles representation. Nevertheless, the orthogonality of the basis enables to construct low order dynamical systems to describe the flow evolution and this convey to POD a great advantage. The principles of such construction is briefly recalled in the next section.

### 2.3.2 Formulation of a low order dynamical system (LODS)

A Galerkin projection enables to rewrite a partial differential equation (PDE) system as a system of ordinary differential equation (ODE). According to this procedure, the functions which define the original equation are projected on a finite dimensional subspace of the phase space (in this case, the subspace generated by the first \( s \) modes).

Following a scheme proposed by Rajaei, Karlsson and Sirovich [35], we project the Navier Stokes equations under the Reynolds decomposition, in order to have an explicit expression for the fluctuating quantities:

\[
\left( \frac{\partial u'}{\partial t} + u' \nabla \bar{u} + \bar{u} \nabla u' + u' \nabla u' - u \nabla u' + \frac{\nabla p'}{\rho} - \nu \nabla^2 (\bar{u} + u') \right) \phi_j = 0. \tag{9}
\]

These equations are obtained by separating flow velocity into mean \( \bar{u} \) and fluctuating \( u' \) parts: \( u = \bar{u} + u' \). Rewriting (9) in terms of POD (7), the resulting equation is a quadratic ODE of order 1. For every \( j \leq s \) modes, the system reads:
\[
\frac{da_k}{dt} = F(a_k) = i_k + \sum_{i=1}^{s} l_{ik} a_i + \sum_{i=1}^{s} \sum_{j=i}^{s} a_i c_{ijk} a_j, \quad k = 1 \cdots s
\]  

(10)

where

\[
l_{ij} = \int_{S} \bar{u} \nabla \phi_i \phi_j ds + \int_{S} \phi_i \nabla \bar{u} \phi_j ds - \int_{S} \frac{1}{Re} \Delta \phi_i \phi_j ds,
\]  

(11)

\[
c_{ijk} = \int_{S} \phi_j \nabla \phi_i \phi_k ds,
\]

(12)

\[
i_k = \int_{S} \nabla p' \phi_k ds - \frac{1}{Re} \int_{S} \Delta \phi_k ds - \sum_{j=1}^{s} \lambda_j \int_{S} \phi_j \nabla \phi_j \phi_k ds.
\]  

(13)

Regarding these expressions, (11) describes the interaction between the mean flow and fluctuating field, it also includes viscous effects from the modes. Nonlinear effects are reported by (12). The independent term (13) takes into account mean flow dissipation, convective contribution of the modes and the pressure field influence.

Boundary conditions and symmetry make the pressure term vanish in particular case of wake flow. As a matter of fact, each of the modes functions satisfies the continuity equation, to give:

\[
\int_{S} \nabla p' \phi_k ds = \oint_{C} p' \phi_k dc,
\]

where \(C\) is the boundary curve of domain \(S\). Works of Deane et. al [19] and Noack et. al. [37] demonstrated that for wake flow configuration, the latter expression is negligible compared to the other terms. Nevertheless, the inclusion of \(p'\) term can be modeled through an additional quadratic expression of the temporal modes \(a\). A noisy version of the momentum equation (10) can also be considered to model the influence of this term. Such a situation is the core of the POD-assimilation technique proposed in [18] and that is briefly described in the third section of this document.

Direct calculation of each term of the system (10) can be avoided by using polynomial identification techniques [4, 32]. This consists in estimating the polynomial coefficients through a least squares fitting. Provided each observation \(a_i\) and its derivative \(\dot{a}_i\), we can write (10) as the solution of a linear system. This method presents the great advantage to avoid the accurate
computation of the basis functions spatial derivatives that are required to construct the projected model (11-13). It can be pointed out also, that even if closed pairs of observation \( a_i(t_k) \) and \( a_i(t_{k+1}) \) are needed to compute the temporal derivative (10), those pairs are not required to be correlated.

In section 3 of this document, we will show how this technique can be greatly improved and how the parameters of this reduced dynamical system together with the initial condition of the temporal modal coefficients can be estimated from an optimal control problem. This process will lead us to propose a technique allowing to characterize the evolution of a POD representation from experimental velocity fields.

### 2.4 Dense non parametric representation of fluid flows

The previous reduced representation allows only to describe the large motion scales of the flows. Smaller scales can only be described through dense representations lying on fine discrete grids with small mesh size. These representations are common and consist in dense vector fields, or alternatively in divergence and vorticity maps. On the image grid, such representation enables theoretically to represent motion scales corresponding to the pixel level. Nevertheless, in reality, the associated estimation process used to estimate dense representations from consecutive images limits actually their true representative finest scale. Only rougher scales level than the pixel level are indeed achievable. These limitations are due to additional smoothness constraints used either in PIV techniques or in optical flow estimator.

Different techniques for the estimation of such dense representations have been studied in the workpackage 2. These methods allow to estimate from two consecutive images, vector fields, potential functions or vorticity and divergence maps. They belong to different families and requires various techniques. Nevertheless, all of these techniques relies on a temporal variation model of the intensity function and an explicit smoothing \textit{a priori} of the velocity fields. This \textit{a priori} allows to solve this difficult inverse problem but imposes questionable smoothing on the solution. Even if sound smoothing devoted to the recovering of typical flow fields can be set such smoothing functional can hardly be related to physical general law on the spatial topology of the flow and as a consequence constitutes at the end always \textit{adhoc} spatial priors.

Besides, fluid flow dedicated motion estimators usually rely on specialized instantaneous data models and on kinematic constraints. They provide independent instantaneous motion field measurements at each frame instants. Each estimation along the image sequence is as a consequence independent from each other. No consistency on the whole motion field trajectories...
can be guaranty. In order to go further, and introduce a physically reliable temporal consistency by the way of appropriate dynamical constraints related to the observed fluid flow, it is essential to consider the motion estimation issue as a dynamic estimation process along a time resolved sequence of image frames.

The introduction of evolution law related to the flow dynamics and the tracking of these dense representations from image sequence will give us an alternative way of estimating motion fields. We will show that such techniques allows to recover much more accurately smaller scales of the motion.

3 Tracking of salient fluid flow structures

A tracking process aims at recovering the whole trajectory of a state variable from noisy and incomplete observations. To that end and to be robust, the available tracking frameworks combine the set of available measurements with a dynamical law of the state variable. Two kinds of methodologies can be distinguished: stochastic filtering or variational data assimilation framework. The first family of methods seeks to estimate the probability distribution of the state variable given the whole set of past measurements. These techniques have the advantage to enable to estimate an incertitude on the solution. Recursive solutions exist and enable the Constitution of on-line techniques which are well adapted to real time control problem. The state variable dimension constitutes nevertheless their Achille’s heel. As in the general case sampling must be drawn from a distribution related to the state variable space, spaces of high dimension can not be straightforwardly considered. The second type of techniques issues from optimal control theory. They are deterministic methods. They allow to estimate a single trajectory of the state variable and no uncertainty estimation is possible. Nevertheless they are naturally well suited to state spaces of great dimension.

Considering the drawbacks and the advantages of these two methodologies, we tried to explore in this work both of them. In the following section, we will describe the variants of these techniques we used and we will present also several applications that have been developed within the FLUID project.

3.1 Recursive estimation through stochastic filtering

We present in this section the stochastic filtering problem in continuous time with discrete observations.
Stochastic filters constitute procedures to estimate the posterior pdfs, \( p(x_k|z_{1:k}) \), of a state variable of interest, given all the measurements until time \( k \) and assuming the sequence \( x_{0:n} = \{x_0, ..., x_n\} \) is a Markov process, with initial distribution \( p(x_0) \).

At each time instant \( k \), the measurement equation relates the observation \( z_k \) to the state \( x_{t=k} \). The corresponding state space model is described by:

\[
\begin{align*}
    dx_t &= f(x_t)dt + \sigma(t)dB_t, \\
    z_k &= g(x_{t=k}) + v_k,
\end{align*}
\]

where \( B_t \) is a Brownian motion and \( v_k \) is a noise variable. The functions \( f \) and \( g \) are non-linear in the general case.

Assuming there exists a transition distribution \( p(x_t|\mathcal{F}(x_{r\leq t})) \) (which should formally be written as \( p(x_t|\mathcal{F}(x_{r\leq t})) \)) where, \( \mathcal{F} \), denotes the filtration generated by the initial state and the Brownian motion up to time \( t = k \), the inference of the posterior pdf may be obtained in two stages:

- Given \( p(x_{k-1}|z_{1:k-1}) \), the prediction step uses the transition distribution \( p(x_t|\mathcal{F}(x_{r<k})) \) to make a first approximation of the next state:
  \[
  p(x_{t=k}|z_{1:k-1}) = \int p(x_{t=k}|x_{r=k-1})p(x_{k-1}|z_{1:k-1})dx_{r=k-1}.
  \]

- The likelihood \( p(z_k|x_{t=k}) \) of a new observation \( z_k \) is then used to update the posterior pdf at instant \( k \):
  \[
  p(x_{t=k}|z_{1:k}) = \frac{p(z_k|x_{t=k})p(x_{t=k}|z_{1:k-1})}{\int p(z_k|x_{t=k})p(x_{t=k}|z_{1:k-1})dx_t}.
  \]

Considering particular forms of either the dynamics or the measurement equation lead to simplifications of the filtering problem. We present in the following different cases.

### 3.1.1 Kalman Filter

In the case of a linear Gaussian model composed of a linear dynamics

\[
x_t = Fx_{t-1} + v_t,
\]

and a linear measurement expressed as:

\[
z_t = Hx_t + \epsilon_t,
\]
where $\epsilon_t$ and $v_t$ denotes white Gaussian noises of covariance $R$ and $Q$ respectively, and $H$ and $F$ are respectively $n \times n$ and a $p \times n$ matrices. In that case the posterior distribution is Gaussian and fully characterized by its two first moments. These moments can be analytically computed. The corresponding recursive updating equations constitutes the Kalman-Bucy filter.

The Kalman filter’s formulas [31] are given with respect to an initial distribution $p(x_0|z_0)$, described by a Gaussian of mean $E(x_0)$ and a variance $\Sigma_0$. The filter is updated at successive times, $t$, through the two following steps:

- **Prediction:**

\[
E(x_t|x_{t-1}) = F E(x_{t-1}),
\]

\[
\Sigma_{x_t|x_{t-1}} = F \Sigma_{x_{t-1}} F^T + Q.
\]

- **Correction:**

\[
K = \Sigma_{x_t|x_{t-1}} H^T (H \Sigma_{x_t|x_{t-1}} H^T + R)^{-1},
\]

\[
E(x_t) = E(x_t|x_{t-1}) + K (z_t - HE(x_t|x_{t-1})),
\]

\[
\Sigma_{x_t} = (I - KH) \Sigma_{x_t|x_{t-1}}.
\]

The expectation $E(x_t)$ and the covariance $\Sigma_{x_t}$ define the Gaussian filtering distribution at time $t$. The matrix $K$ that takes place in the mean and covariance updates defines the so called Kalman gain.

This technique has been used by the AMI group to define a point trajectory estimation technique. The state variable consists in the location of a point, the dynamical law is given by an affine motion model whereas noisy correlations constitutes the measurements. Let us note that when the system is only conditionally Gaussian with respect to the image data, then very similar update rules can be properly defined [2]. Nevertheless it is not anymore optimal, and only a conditional filtering distribution may be estimated.

### 3.1.2 Ensemble Kalman Filter

An extension of the Kalman filter, adapted to non linear dynamics associated to linear measurements model, called *ensemble Kalman filter*, has been proposed for geophysical flow analysis by Evensen [21] (see also [5, 22, 23]). This method relies on a Monte Carlo sampling of the filtering law. The different probability distributions are now described by samples of the law. The initial law is sampled by $N$ members (or particles) $x_0(i), i = 1 \ldots N$. 

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The prediction step consists in propagating the ensemble of particles through the non linear dynamics (including the noise simulation) in order to obtain the predicted particles, or forecast ensemble, denoted by \( x_i^{(t)} \). The EnKF relies on a Gaussian approximation of the particles nest, in order to apply Kalman’s formulas in the following way.

The mean particle is firstly defined by:

\[
\bar{x}_{t|t-1} = \frac{1}{N} \sum_{i=1}^{N} x_i^{(t)}.
\]  

(22)

The Ensemble covariance matrix \( P_{t|t-1}^e \) is also empirically defined:

\[
P_{t|t-1}^e = \frac{1}{N-1} \sum_{i=1}^{N} \left( x_i^{(t)} - \bar{x}_{t|t-1} \right) \left( x_i^{(t)} - \bar{x}_{t|t-1} \right)^T.
\]  

(23)

Similarly to the Gaussian case presented in the previous section, an ensemble Kalman gain \( K_t^e \) is also defined from the ensemble variance:

\[
K^e = P_{t|t-1}^e H^T (H P_{t|t-1}^e H^T + R)^{-1}.
\]  

(24)

The corrective step consists then to associate each particle, \( x_i^{(t)} \), to a perturbed observation \( z_t + \epsilon^{(i)} \), and to apply the Kalman filter corrective steps to obtain the new states of the ensemble \( x_i^{(t)} \), defined as:

\[
x_i^{(t)} = x_i^{(t)} + K^e (z_t + \epsilon^{(i)} - H x_i^{(t)}).
\]  

(25)

In case of high dimensional state spaces, the measurement noise covariance \( R \) is usually replaced by an empirical covariance \( R^e \) computed from the samples \( \epsilon^{(i)} \) of the measurements noise law. In this case, the gain \( K^e \) and the ensemble covariance \( P_{t|t-1}^e \) are not directly computed, but are simply represented by the ensemble (see [22, 23] for further details). In this procedure, ensemble covariance matrices, whose dimension is the size of the state space to the square, are neither explicitly computed nor stored. This point is crucial when dealing with high dimensional state spaces such as those involved in geophysical sciences.

The new ensemble obtained after the analysis is then used as an initialization for the next time step.

3.1.3 Particle filter

Particle filtering can be applied to much general case, involving non linear likelihood, non additive and non Gaussian noises. These techniques propose to implement recursively a Monte-Carlo approximation of the sought density \( p(x_{t=k}|z_{1:k}) \). This approximation consists in a finite
The weighted sum of $N$ Diracs centered on hypothesized locations in the state space – called particles – of the initial system $x_0$. At each particle, $x_k^{(i)} (i = 1 : N)$, is assigned a weight $w_k^{(i)}$ describing its relevance. This approximation reads then:

$$p(x_{t=k} | z_{1:k}) \approx \sum_{i=1:N} w_k^{(i)} \delta_{x_k^{(i)}}(x_k).$$  \hspace{4cm} (26)

Assuming that the approximation of $p(x_{t=k-1} | z_{1:k-1})$ is known, the recursive implementation of the filtering distribution is done by propagating the swarm of weighted particles $\{x_{i}^{(k)}_{t=k-1}, w_{k-1}^{(i)}\}_{i=1:N}$. At each time instant (or iteration), the set of new particles $\{x_t^{(i)}\}_{i=1,N}$ is drawn from an approximation of the true distribution $p(x_{t=k} | z_{1:k})$, called the importance function and denoted $\pi(x_t | x_{0:t-1}^{(i)}, y_{1:k})$. The closer the approximation to the true distribution, the more efficient the filter. The importance weights, $w_k^{(i)}$, account for the deviation w.r.t. the unknown true distribution. To maintain a consistent sample, the importance weights are updated according to a recursive evaluation as the new measurement $z_k$ becomes available:

$$u_k^{(i)} \propto w_k^{(i)} \frac{p(z_k | x_k^{(i)}_{t=k}) p(x_k^{(i)}_{t=k} | x_{t<k}^{(i)})}{\pi(x_t | x_{0:t-1}^{(i)}, y_{1:k})}, \quad \sum_{i=1:N} u_k^{(i)} = 1.$$  \hspace{4cm} (27)

Different choices are possible for this proposal density [1]. The most common one consists to set the proposal distribution to the dynamics:

$$\pi(x_t | x_{0:t-1}^{(i)}, y_{1:k}) = p(x_t | x_{t-1}^{(i)}).$$

In this case the weight computation is greatly simplified by factorizing the expression with the data likelihood $p(z_t | x_t^{(i)})$. This version of the particle filter is known as the bootstrap filter.

**Vorticity Bootstrap Filter** Such an implementation has been used to define a filter for the tracking of vorticity [15, 16]. In this work a stochastic 2D formulation of the velocity vorticity formulation of the Navier Stokes is coupled with a non linear image reconstruction error. More precisely, the state vector is composed of a set of vortex particle radial basis function. The vector $X = (x_i, i = 1, \cdots, p)^T$ represents the set of vortex particle locations. A coefficient, $\gamma_i$, encoding the strength of each elementary vortices is associated to each basis function.

A stochastic interpretation of the vorticity transport equation may be defined through the following stochastic differential equation [7]:

$$dX_t = w(X_t) dt + \sigma dB_t,$$  \hspace{4cm} (28)
where $dB$ is a $2p$-dimensional Brownian motion with independent components, and variance $\sigma = \sqrt{2\nu} I_{2p}$. The evolution of the vortex set, $X$, between two frame instants $k$ and $k+1$ and for discretization steps $\Delta t$ is represented by the following Markov transition equation:

$$p(X_j^k | X_{j-\Delta t}^k) \sim N(X_{j-\Delta t}^k + w(X_j^k)\Delta t, 2\nu \Delta t I_{2p}),$$

(29)

where $I_{2p}$ denotes the $2p \times 2p$ identity matrix.

An error reconstruction measurement, $z_k$, can be constructed from the pair of images $(I_k, I_{k+1})$. This provides us a likelihood $p(z_k | X_k)$ of the data. The transition probability transition and a likelihood distribution authorize to settle a Monte-Carlo probabilistic technique (see [27] for a review on this technique) enabling to get access to the conditional distribution $p(X_k | z_{1:k})$ of the vortex set given the set of measurements from the initial time and up to time $k$. This approach is described in the following section.

The basic filtering procedure consists roughly in our case to sample a set of individuals from the stochastic differential equations (28), each individual of the sample (here a set of vortex particles) is then weighted by their likelihood measure. This provide an empirical posterior distribution from which a new sample can be drawn and so on until the end of the sequence.

As samples must be drawn in a high dimensional state space, the efficiency of this method is limited to state features of limited size. Only a relatively small number of vortex particle can be thus handled. On the other hand, as such an approach is intrinsically a Lagrangian approach no complex discretisation scheme are needed for the advective term of the corresponding Eulerian formulation. Such a technique reveals to be attractive to track the large scales of the motion. In figure 2 we plotted several velocity fields obtained on a 60 frames real world sequence showing a laser tomography of a vortex launch at tip of airplane wing. In this sequence the flow has been seeded with smoke.

A further description of the tracker, and additional experimental results for meteorological application can be found in [15].

**Extension of the Ensemble Kalman Filter for vorticity tracking** An extension of the Ensemble kalman filter has been very recently defined for vorticity tracking from velocity fields measurements. Like the original ensemble Kalman filter its use is restricted to systems composed of a non linear dynamics and a linear Gaussian measurement model. This extension has been defined through a definition of the proposal distribution based on the ensemble Kalman filter (EnKF) principles [26]. The insertion of the EnKF updating rules within the particle filter
Figure 2: Sequence of motion fields obtained by stochastic filtering on a laser tomography of a vortex launched at tip of an airplane wing - image sequence courtesy of ONERA.
scheme allows us to make the resulting filter exact (up to the sampling) and thus to increase its accuracy.

The expression 25 defines the displacement of the ensemble members; it can be rewritten as:

$$x_t^{(i)} = (1 - K^e H)f(x_{t-1}^{(i)}) + K^e z_t + K^e \epsilon_t^{(i)} + (1 - K^e H)v_t^{(i)}$$

$$= \mu_t^{(i)} + \gamma_t^{(i)},$$

where the mean \(\mu_t^{(i)} = (1 - K^e H)f(x_{t-1}^{(i)}) + K^e z_t\) of \(x_t^{(i)}\) has been introduced. The term \(\gamma_t^{(i)}\) is the random part of the expression. It is a weighted sum of two Gaussian variables, and thus also follows a Gaussian distribution with mean zero and variance \(\Sigma_t^e\):

$$\gamma_t^{(i)} \sim \mathcal{N}(0, \Sigma_t^e),$$

with \(\Sigma_t^e = (1 - K^e H)Q(1 - K^e H)^T + K^e RK^e T\),

Here \(\mathcal{N}(\mu, \Sigma)\) denotes a Gaussian distribution with mean \(\mu\) and variance \(\Sigma\) and constitutes the importance function on which we are going to rely on. In what follows, a notation \(\mathcal{N}(x; \mu, \Sigma)\) will denote the value of this Gaussian at \(x\).

In order to make the estimation of the filtering distribution exact (up to the sampling), the formulas of the particle filter are applied. As a consequence, each member of the ensemble must be weighted at each time, \(t\), with a weight, \(w_t^{(i)}\), defined iteratively by:

$$w_t^{(i)} \propto w_{t-1}^{(i)} \frac{p(z_t | x_{t|t}^{(i)})p(x_{t|t}^{(i)} | x_{t-1|t-1}^{(i)})}{\mathcal{N}(x_{t|t}^{(i)} - \mu_t^{(i)}; 0, \Sigma_t^e)},$$

and \(\sum_{i=1}^{N} w_t^{(i)} = 1\).

The proposal distribution depends on the ensemble Kalman gain estimated from the mean and the covariance of the predicted particles \(x_{t|t-1}^{(i)}\). But the ensemble members are weighted and the formulas (22) and (23) must be adapted in consequence to take into account these weights. The ensemble Kalman gain and the new propagation law of the particles at each time step are then computed from these weighted particles.

Let us note that even if the ensemble Kalman’s gain \(K^e\) is not evaluated from the particles, we can give him some particular values and recover some well known filters:

- By enforcing \(K^e\) to zero, the Bootstrap filter is obtained.
- A filter with the propagation law centered on the future observation can be built by enforcing \(K^e\) to 1. The weights updating will be then linked to the dynamical law between the positions of a particles at times \(t - 1\) and \(t\).
• Finally, if we chose the gain as $K^e = QH^T(HQH^T + R)^{-1}$, the optimal importance particle filter is recovered [1].

Further details on the implementation of this new filter are described in [26]. It is also demonstrated how the WEnKF increases the performances of the EnKF and leads to much lower computation time. This filter has been applied for the tracking of vorticity of a 2D turbulent fluid motion. The dynamical model used corresponds to a discretization of the 2D incompressible vorticity-velocity formulation of the Navier-Stokes equations:

$$\xi_t = \xi_{t-1} - \nabla \xi_{t-1} \cdot \mathbf{w}_{t-1} + \frac{1}{Re} \Delta \xi_{t-1} + \nu_t$$  \hspace{1cm} (33)

where $\nu_t$ is a white Gaussian noise of variance $Q$ and $Re$ is the Reynolds number representing the turbulence of the flow. A linear measurements model (16) is still used. The respective uncertainty of the dynamics and of the measurements are computed from pseudo-random fields simulated according to the Evensen’s method, developed in the Appendix of the paper [21].

The measurements are obtained by computing the vorticity with a robust optical flow method [11]. The vorticity obtained by this method is used as measurement for the Ensemble Kalman filters. Let us remark that, in this application, the noise corresponding to such an observations is non Gaussian and unknown.

The figure 3 shows an example of the kind of results obtained with the Wenkf techniques. This experiments has been conducted a synthetic image sequence created through a Direct Numerical Simulation of the vorticity-velocity equation, with a Reynolds number $Re = 4000$ and a forcing term on the vorticity which corresponds to a model noise $\nu_t$ with covariance $Q = 0.005$. This data sequence is further described in the FLUID deliverable 1.2 entitled "Report on production and diffusion of fluid mechanics images and data".

As can be observed the small scales of the noisy observations seems quite well restored. This can be check on figure 3.1.3 which compares the spectrum of the row average vorticity energy of the ground truth and the one obtained from WEnKF. The large scales of the motion are not fully recovered with the WEnKF. The large scales representation of the observations are even closer to the ground truth. At the opposite, the small motion scales are well recovered with this technique and of much better quality.

An assessment of the techniques on several examples demonstrated that the modified version of the ensemble Kalman filter led in almost all the case to better results and also to a significant decrease of the computational time [26].

This new ensemble Kalman filtering technique seems to have a great potentiality. It overcomes the limitation of the particle filter with regard to high dimensional state spaces. In the
Figure 3: Results obtained for the WEnKF on turbulent 2D flows. From left to right: true vorticity map, observed vorticity map, recovered vorticity.

Figure 4: Vorticity spectrum of the 2D DNS turbulent flow (black), vorticity spectrum of the measurements (blue), vorticity spectrum of the assimilated motion field with the WEnKF.
one hand, it allows to track quite nicely the small motion scales. The large scales seems less well reconstructed than in the particle filtering case. Nevertheless, it must be pointed out that the two vorticity tracking can not directly be compared as they do not rely on the same measurements and on the same dynamical noise. These two filters should be more intensively compared to draw clear conclusions. This comparison should be done on more 3D realistic problems, in order to clearly see the potentiality of these filters regarding applications in fluid mechanics or in environmental sciences.

3.2 Global estimation control approach

In this section we present a variational formalism which enables also to filter noisy and incomplete data by a dynamical model. This framework issuing from optimal control theory has been popularized in geophysical sciences and is known as variational assimilation [20, 34]. In contrast with particle filtering, variational assimilation techniques have the advantage to enable a natural handling of high dimensional state spaces. Before presenting further the adaptation of such a framework to fluid motion tracking, let us describe its general principles.

The problem we are dealing with consists in recovering a system’s state \( X(x, t) \) driven by a dynamical law and observed through noisy and possibly incomplete measurements. The measurements (also called observations) are assumed to be available only at discrete time. This is formalized, for any location, \( x \), at time, \( t \in [t_0, t_f] \), by the system:

\[
\frac{\partial X}{\partial t}(x, t) + M(X(x, t), u(t)) = 0
\]

\[
X(x, t_0) = X_0(x) + \epsilon_n(x),
\]

where \( M \) is a non-linear dynamic operator depending on a control parameter \( u(t) \). The term, \( X_0 \), is the initial vector at time, \( t_0 \), and, \( \epsilon_n \), is an (unknown) additive control variable on the initial condition. Besides, we assume that measurements of the unknown state (also called observations), \( Y \in \mathcal{O} \), are available. These observations are related to the state variable through the non-linear operator, \( \mathbb{H} \).

The goal consists to find an optimal control of lowest energy that leads to a minimum discrepancy between the measurements and the state variable. This can be expressed through the minimization of cost function \( J \):

\[
J(u, \epsilon_n) = \frac{1}{2} \int_{t_0}^{t_f} \| Y - \mathbb{H}(X(u(t), \epsilon_n, t)) \|^2_{R^{-1}} dt
+ \frac{1}{2} \| \epsilon_n \|^2_{B^{-1}} + \frac{1}{2} \int_{t_0}^{t_f} \| u(t) - u_0 \|^2_{P^{-1}} dt,
\]

(36)
where $u_0$ denotes some expected value of the parameter. The norms $\| \cdot \|_{R^{-1}}, \| \cdot \|_{B^{-1}}$ and $\| \cdot \|_{F^{-1}}$ are induced norms of the inner products $\langle \cdot , \cdot \rangle_{\mathcal{O}}, \langle \cdot , \cdot \rangle_{\mathcal{V}}, \langle \cdot , \cdot \rangle_{\mathcal{U}}$ associated to Hilbert spaces $\mathcal{O}, \mathcal{V}, \mathcal{U}$ of the measurements, the state variable and the control variable respectively; $R, B$ and $F$ are endomorphism (called covariances matrices) of the observation space and of the state space. They are related respectively to the uncertainty associated to observations, the initial condition of the state variable and the expected value of the control variable. In our applications, these covariance matrices have been generally defined as diagonal or band-diagonal matrices. In order to compute the gradient of this functional, we assume that $X(u(t), \epsilon_n; t)$ depends continuously on $(u(t), \epsilon_n)$ and is differentiable with respect to $(u(t), \epsilon_n)$ for all $t \in [t_0, t_f]$.

A first brute force approach would consist in computing the functional gradient through finite differences:

$$
\nabla_v J \simeq \lim_{\epsilon \to 0} \frac{J(v + \epsilon e_k) - J(v)}{\epsilon},
$$

where $\epsilon \in \mathbb{R}$ is an infinitesimal perturbation and $\{ e_k, k = 1, \ldots, N \}$ denotes the unitary basis vectors of the control space of dimension $N$. Such a computation is impractical for control space of large dimension since it requires $N$ integrations of the evolution model for each required value of the gradient functional.

**Adjoint models** As introduced first in optimal control theory by Lions [34], an elegant solution consists to rely on an adjoint variable driven by an adjoint evolution law. The introduction of this adjoint formulation authorizes the computation of the gradient functional in a single backward integration of an adjoint variable. This framework is widely used in environmental sciences for the analysis of geophysical flows [20, 47].

The adjoint evolution law is given by the following system:

$$
\begin{cases}
- \frac{\partial \lambda}{\partial t}(t) + (\partial_X M)^* \lambda(t) = (\partial_X M)^* R^{-1} (\mathcal{Y} - \mathbb{H}(X))(t), \\
\lambda(t_f) = 0,
\end{cases}
$$

(37)

where the adjoint of the tangent linear operators $(\partial_X M)^* : \mathcal{V} \to \mathcal{V}$ and $((\partial_u M))^* : \mathcal{V} \to U$ have been introduced. Injecting this relation into the cost-function gradient allows to obtain simple expression in term of the adjoint variable trajectory of the functional gradient components. C cancelling these relations enable the definition of gradient descent strategies. The adjoint variable is accessible through a forward integration of the state dynamics (34-35) and a backward integration of the adjoint variable dynamics (37). The overall optimal control process can then be schematically summarized as follows:
1. Set an initial condition: $X(t_0) = X_0$

2. From $X(t_0)$, compute $X(t)$ with the forward integration of relation (34)

3. Compute the adjoint variable $\lambda(t)$ with the backward integration of relation (37)

4. Update the initial value $X(t_0)$ and the control $u(t)$ from the functional gradient expressions

5. Loop to step 2 until convergence

Such a formulation endows a practical framework for the tracking of very large state spaces. Nevertheless, unless using a small temporal integration window of two frames, the approach is intrinsically a batch technique. Unlike stochastic filtering techniques no recursive approach exists for such a procedure.

This formalism has been exploited for different tracking issues related to the analysis of fluid flows from image sequences. The first one concerns the setup of optical flow estimator through a 2D vorticity velocity formulation of Navier-Stokes equation or a simplified vorticity divergence form of the Shallow water dynamical model. The second application consists in a joint tracking of closed curves and their motions. The third one aims at tracking layered atmospheric motions. This last application is also related to the Workpackage 4. For each model we present briefly the dynamics, the associated image measurement used, the ingredients of the assimilation process and some experimental results obtained. Further details can be found in the following publication [39, 42, 45, 40, 44, 41]

### 3.2.1 Vorticity-divergence optical flow

In order to consider a tractable expression of the Navier-Stokes equation for the tracking problem, we rely in this work on the 2D vorticity-velocity formulation of the 3D incompressible Navier-Stokes equation, as obtained in a simplified large scale shallow water model:

$$\partial_t \xi + w \cdot \nabla \xi + \xi \zeta - \nu \Delta \xi = 0.$$  \hspace{1cm} (38)

This formulation states roughly that the vorticity is transported by the velocity field and is diffused along time. The vorticity corresponds here to a filtered vorticity with an enstrophy-based subgrid model [36].

Concerning the divergence map, we assume here that it is weak and is advected by the flow and a noise variable that encodes an uncertainty on the model. More precisely we will assume...
that the divergence map is a function of a stochastic process representing a particle position and driven by the following stochastic differential equation:

$$dx_t = w(x_t)dt + \sqrt{2\nu_t}dB_t. \quad (39)$$

This equation states that the particle position is known only up to an uncertainty that grows up linearly with time, $B_t$ is a standard Brownian motion of $\mathbb{R}^2$. The process $x_t$, starts at points, $x_o$.

It can be shown through Ito formula and Kolmogorov’s forward equation, that the expectation at time $t$ of such a function, $\zeta(t,x) = \mathbb{E}[\text{div}w(x_t)]$ obeys to an advection diffusion equation [38]:

$$\partial_t \zeta + \nabla \cdot w + \zeta \text{div} w - \nu \Delta \zeta = 0, \quad \zeta(0, x_o) = \text{div} w(x_o). \quad (40)$$

Assuming that $\text{div}w(x) \approx \zeta(x)$, allows us to write the following imperfect dynamical model for the fluid motion field:

$$\partial_t \left[ \begin{array}{c} \zeta(t) \\ w(t) \cdot \nabla \xi(t) + \zeta(t) \end{array} \right] + \left[ \begin{array}{c} w(t) \cdot \nabla \cdot \xi(t) - \nu \Delta \zeta(t) \\ w(t) \cdot \nabla \cdot \xi(t) - \nu \Delta \zeta(t) \end{array} \right] = p(t). \quad (41)$$

The control function, $p(t)$, is a vector modeling the errors of our evolution law. For geophysical flow it is related to the Coriolis force, to the dissipation of depth layer consider and to the subgrid stress tensor. The discretization of both equation have been considered through Total Variation Diminishing (TVD) scheme and a third-order Runge Kutta scheme [44].

The adjoint evolution model is constructed through a backward integration of the Runge-Kutta scheme used and from the adjoint of the model operator discretization.

As for the measurement model, we rely on the optical-flow equation and the Lucas and Kanade operator. Relying on a brightness consistency assumption ($dI/dt = 0$), and assuming the unknown velocity vector at location, $x$, is constant within some neighbourhood of size $n$, the least squares fit Lucas-Kanade formulation leads to the following linear system:

$$f_n \odot \left[ \begin{array}{cc} E_x^2 & E_x E_y \\ E_x E_y & E_y^2 \end{array} \right] \mathbf{v} \approx -f_n \odot \left[ \begin{array}{c} E_x E_t \\ E_y E_t \end{array} \right], \quad (42)$$

where $E_\bullet = \partial E(x,t)/\partial \bullet$ and $f_n$ denotes a windowing function. In our applications, the system’s state, $\xi(t)$, is related to $\mathbf{v}$ through Biot-Savard relation (1). The available measurements read then:

$$\mathbf{y} = - f_n \odot \left[ \begin{array}{c} f_n * (E_x E_t) \\ f_n * (E_y E_t) \end{array} \right], \quad (43)$$

26
and the observation operator:

\[ \mathbb{H}(\xi(t), t) = \begin{bmatrix}
    f_n * (E_x^2) & f_n * (E_x E_y) \\
    f_n * (E_x E_y) & f_n * (E_y^2)
\end{bmatrix} K_{\epsilon}^{-1} * \xi(t). \] (44)

This observation operator is linear w.r.t the state’s variable \( \xi \). Therefore, its linear tangent operator is itself. The expression of its adjoint is on the other hand not trivial as this operator is expressed through a convolution product with the Green kernel gradient, \( K \). It can be demonstrated that its expression is \( (\partial_\xi \mathbb{H})^* = -\mathbb{H} \).

The following figures illustrate results that have been obtained for the DNS sequence of 2D turbulence. For this 52 frames image sequence, we compare in figure 5 the actual vorticity map, the vorticity corresponding to motion field estimated through an optical flow technique[11], the vorticity obtained after an assimilation of this motion field and the vorticity obtained assimilating directly images. The assimilation of noisy motion fields corresponds to the works presented in [44] and relies on measurements equation that relates directly the sought vorticity and divergence to the noisy observed motion fields.

It can be observed that the assimilation technique based on motion field observations (line d) not only denoises the observations, but also enables to recover small scales structures that were smoothed out in the original velocity fields (line c). A direct assimilation of images (line e) performs even better. For this direct assimilation, the quality of the recovered motion field is improved of nearly 30%.

To give some quantitative evaluation results, we present the root mean square errors in figures 6. In this figure we plotted error measurements corresponding to the motion field supplied by the fluid optical-flow dedicated technique [10], the results obtained after assimilating these observations and the results provided by a direct assimilation of the image data.

It can also immediately be observed that small motion scales are much better recovered in the case of direct assimilation of images. In order to demonstrate this ability more precisely, we pictured in figure 7, a spectral analysis of the energy of the row average vorticity. This curves show the behavior of the different methods with respect to different frequencies of the flow. We can observe that optical flow measurements are not good for the large and the small scales. The assimilation process allows clearly to correct these deficiencies. The direct assimilation fits much accurately the actual curves in the intermediate scales and in the finest scales.

This assimilation technique has been run on Meteosat satellite meteorological sequence of the Infra-red channel. This sequence describes the evolution of the Vince cyclone over the At-
Figure 5: **2D Direct Numerical Simulation, particle sequence.** Motion observations. *a*) Particle images sequence. *b*) True vorticity. *c*) Vorticity of the observed motion fields. *d*) Recovered vorticity after motion field assimilation. *e*) Vorticity estimated by direct assimilation of images.
Figure 6: Particle sequence, Comparison of errors. The root mean square error of the motion fields estimated on the sequence are compared for the three methods: the fluid dedicated optical flow (in blue), assimilation of optical flow observation (in red) and assimilation of image data (in green).

Figure 7: Particle sequence, spectral analysis A spectral analysis of the energy of the row average vorticity is represented in the log-log scale. The actual vorticity (black) is compared to the optical flow vorticity (blue), the vorticity obtained by assimilation of optical flow observation (red) and the one provided by assimilation of image data (green).
lantic Ocean. This is a 20 frames image sequence shot the 10th of October 2005 from 00H00 to 5H00 by the Meteosat Second Generation (MSG) satellite. We show on figure 8, superimposed on the original image data, the motion fields recovered. We can checked from these images that the motion of the cyclone’s eye is very well recovered. This last result illustrate the fact that the estimations provided are coherent with respect to the visualized phenomenon.

Figure 8: Cyclone Vince. Motion fields recovered by direct assimilation of the Infra-red images.

In order to demonstrate the validity of the dynamics and the robustness of the method, we considered a visible channel image sequence of the MSG satellite depicting the same cyclone Vince. This sequence corresponds to 24 hours cyclone’s evolution. Observations are only available during the day and are missing from the 9th of October 19H00 to the 10th of October 8H00. During this time interval, the corresponding covariance matrices $R(t)$ are automatically set to zero as no photometric contrast are observed. The assimilation results are plotted in terms of velocity fields and are superimposed on the infra-red images in the figure 9 in order to assess the quality of the measurements. We can evaluate visually the quality of the estimations. As
the cyclone increased its speed during night time, the estimation is not perfect. It nevertheless demonstrates that the approximate motion dynamics used in the previous example reveals to be quite good as it allows to reconstructed the cyclone dynamics despite a very large occlusion.

Further results on DNS data and on meteorological sequences are described in [42]. This technique has been also extended for the joint tracking of curves and its underlying motion. We describe this extension in the following section.

3.2.2 Joint curves and motion tracking

In this work, the purpose was to design a technique allowing to handle the tracking of a closed curve transported by a motion field. We wanted in particular to track photometric iso-lines of fluid phenomenon. In order to authorize topological changes of the curves of interest we relied on an implicit level set representation of curves. Within that framework, the curve $\Gamma(t)$ enclosing the target to track is implicitly described by the zero level set of a scalar function $\phi(x, t) : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R} : \Gamma(t) = \{ x \in \Omega \mid \phi(x, t) = 0 \}.$

Curve evolution law In order to define a dynamics for the unknown surface, we will assume that the curve is propagated at each frame instant by a velocity field, $w(x, t) = [u(x, t), v(x, t)]^T,$ and diffuses according to a mean curvature motion. This dynamics is assumed to be valid up to an additive control function $p(x, t)$:

$$\partial_t \phi + (w \cdot n - \varepsilon \kappa) \|
\phi\| = p, \tag{45}$$

The curvature and the normal are directly given in term of surface gradient: $\kappa = \text{div}(\nabla \phi/\|
\nabla \phi\|)$ and $n = \nabla \phi/\|
\nabla \phi\|.$ The corresponding semi-implicit discretization of the curve evolution law we used is described in [43, 45]. The motion field transporting the curve is given by the previous vorticity-divergence assimilation scheme.

Curve measurement equation To link the image data to the unknown surface variable we rely on a measurement modeling that involves local probability distributions of the intensity function. This model compares at each point of the image domain a local photometric histogram to two global probability density functions $\rho_o$ and $\rho_b$ modeling respectively the object and background intensity distribution. These two distributions are assumed to be estimated from the initial location of the target object. The measurements equation we propose reads:

$$F(\phi, I)(x, t) = [1 - d_B(\rho_{V_x}, \rho_o)]^2 1_{\phi(x) < 0} + [1 - d_B(\rho_{V_x}, \rho_b)]^2 1_{\phi(x) \geq 0} = \epsilon(x, t), \tag{46}$$

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Figure 9: **Vince cyclone tracked by night.** The motion fields obtained from the assimilation computed with the *visible images* are superimposed: (a) on the *visible channel images*, (b) on the *infra-red channel images*. There is a data occlusion from the 9th of October at 19H00 to the 10th of October at 8H00. As there is no observations during night-time on the visible channel, the velocities estimations of the cyclone eye are not perfect.
Figure 10: First row: tracking of a closed curve delineating a cyclone observed over the Indian ocean. Second row: vorticity assimilated from noisy vector fields. Third row: noisy vorticity maps extracted from a noisy local optical flow measurements

where $\epsilon$ is a Gaussian variable, $\rho_{\nu}$ denotes the local intensity probability density function and $d_B$ is the Bhattacharya probability density distance measure defined as:

$$d_B(\rho_1, \rho_2) = \int_0^{255} \sqrt{\rho_1(z)\rho_2(z)}dz.$$  

Let us note that by replacing the densities with intensity average, we retrieve the Chan and Vese functional proposed for image segmentation[6].

The figure 10 describes the results obtained for the tracking of a cyclone observed over the Indian ocean from the infra-red channel of the Meteosat satellite. On this figure we show the closed curve trajectory but also the assimilated vorticity.

Further results on meteorological sequences and on DNS data can be found in [9, 43]. In the next section, we show how the assimilation of fluid motion can be extended to recover horizontal components of atmospheric flows from a Shallow water model and difference of pressure images.
3.2.3 Shallow water optical flow

This study aims at recovering stratified atmospheric wind fields from satellite images. In the same way as the previous one, it is based on a variationnal assimilation approach and a large scale Shallow-Water dynamical model. Nevertheless, the model relies now on a layering of the atmosphere. Such layering of atmospheric flow is valid in the limit of horizontal scales much greater than the vertical scale height, thus roughly for horizontal scales greater than 100 km. In order to make the layering assumption valid in the case of satellite images of kilometer order, low resolution observations relevant to the model grid scale are considered. In the proposed model the 3D space is decomposed into sheets of variable thickness, corresponding to pressure layers. Analysis based on such decomposition presents the main advantage of operating at different atmospheric pressure ranges and avoids the mix of heterogeneous observations.

Layer decomposition The 3D space decomposition in $K$ layers is defined as follow. The $k$-th layer corresponds to the volume lying in between an upper surface, $s_{k+1}$, and a lower surface, $s_k$. These surfaces, $s_j$, are defined by the height of top of clouds belonging to the $j$-th layer. They are thus defined only in areas where there exists clouds belonging to this layer, and remains undefined elsewhere. The membership of top of clouds to the different layers is determined by cloud classification maps. Such classifications which are based on thresholds of top of cloud pressure, are routinely provided by the EUMETSAT consortium, the European agency which supplies the METEOSAT satellite data. This decomposition is illustrated on figure 11.

Figure 11: Illustration of the multi-layer model & sparse transmittance maps.
Sparse pressure difference observations  Top of cloud pressure images are also routinely provided by the EUMETSAT consortium. They are derived from a radiative transfer model using ancillary data obtained by analysis or short term forecasts. Multi-channel techniques enable the determination of the pressure at the top of semi-transparent clouds. In this work we rely on on measurements \( h_{k}^{\text{obs}} \) given by differences of top of cloud pressure images. For a given layers such images are defined as:

\[
 h_{k}^{\text{obs}} = \begin{cases} 
 \tilde{p}^k - p_{\partial k} & \text{in } C^k \\
 0 & \text{in } C^k. 
\end{cases} 
\]  

We denote by \( C^k \) the classification map corresponding to the \( k \)-th pressure layer. The union of top of cloud pressure image segments is denoted \( p_{\partial k} \), it corresponds to pressure observed at the upper boundary surface of layer \( k \). The pressure on the inferior boundary layers denoted, \( \tilde{p}^k \), is assumed to be given by the upper boundary surface of the underneath layer. The missing data on the inferior surface are approximated by an average pressure value.

Shallow water model  In order to provide a dynamical model for the previous pressure difference observations, we use the shallow-water approximation (horizontal motion much greater than vertical motion) derived under the assumption of layer incompressibility (layers are characterized by mean densities \( \rho^k \)). Let us remark that the shallow- water approximation is valid for mesoscale analysis in a layered atmosphere. As friction components can be neglected, the vertical integration of the momentum equation between boundaries \( z^k \) and \( z^{k+1} \) yields for the \( k \)-th layer to the equation (dropping the index \( k \) for sake of simplicity):

\[
 \begin{cases} 
 \frac{\partial h}{\partial t} + \text{div}(h\mathbf{v}) &= 0 \\
 \frac{\partial (q)}{\partial t} + \text{div}(\frac{1}{\rho} \mathbf{q} \mathbf{q}^T) + \frac{1}{\rho} \nabla_x (h)^2 &= 0,
\end{cases} 
\]  

where the involved quantities are defined as:

\[
\begin{align*}
 h &= p|_{z=z^k} - p|_{z=z^{k+1}} \\
 \mathbf{v} &= (u, v) = \frac{1}{h} \int_{p|_{z=z^k}}^{p|_{z=z^{k+1}}} \mathbf{v} dp \\
 \mathbf{q} &= h \mathbf{v}.
\end{align*}
\]  

We have defined all the ingredients required for the assimilation system for the tracking of the pressure difference \( h^k \) and the average velocity \( q^k \) of the \( k \) layer. The dynamic law of the state variable are given by the shallow-water model described previously. The available observations
are composed of the pressure difference maps \( h^k_{obs} \). Given two initial conditions \( h^k|_{t=0} = h^k_0 + \epsilon_h \) and \( q^k|_{t=0} = q^k_0 + \epsilon_q \), we thus seek the optimal control on the initial conditions minimizing the cost function:

\[
J_k(\epsilon_h^k, \epsilon_q^k) = \int_{t_0}^{t_f} \| h^k_{obs} - h^k(\epsilon_h^k, \epsilon_q^k) \|^2_{\mathbb{R}^h} + \| \epsilon_h^k \|^2_{B_h} + \| \epsilon_q^k \|^2_{B_q} dt. \quad (51)
\]

The initial pressure is initialized by a flat pressure map corresponding to the average of the observed pressure on the given layer and the initial horizontal wind is given by a motion estimator dedicated to the estimation of atmospheric layer [29]. The dynamical model has been discretized with adapted numerical schemes [41].

The figure 3.2.3 presents some results obtained for three atmospheric layers. This benchmark data corresponds to a sequence of 10 METEOSAT Second Generation (MSG) images, showing top of cloud pressures with a corresponding cloud classification sequence. The 1024 x 1024 pixel images cover an area over the north Atlantic Ocean during part of one day (5-June-2004), at a rate of one image every 15 minutes. The spatial resolution is 3 kilometers at the center of the whole Earth image disk. Clouds from a cloud-classification were used to segment images into 3 broad layers, at low, intermediate and high altitude. In order to make the layering assumption valid, low resolution observations on an image grid of 128x128 pixels are obtained by adequate smoothing and sub-sampling of each data layer.

Interested readers may find a comparative study between this assimilation model and the previous one based on vorticity divergence formulation in [9]. This comparison leads on the same pressure images shows that both techniques lead to very comparable results. The direct image assimilation technique based on approximate Shallow-Water model enables to recover more finer motion scales than the assimilation based on the perfect shallow-water model. This seems to confirm that the coupling of image data with inexact dynamical model may be an interesting approach to recover coherent fluid flow velocity fields.

As shown through these different examples, variational assimilation has allowed us to reconstruct nicely experimental flows from image sequences. However this techniques has been implemented only on quasi 2D flows and the question of their efficiency to cope with more complex situations and 3D flows remains. It would be in particular interesting to address the case of large eddies models with images depicting finer motion scales of the phenomenon.

Besides, the introduction of control variables within the dynamics enabled to adapt approximate dynamical models to image data. However, in the previous study the control variable did not have particular physical meaning. It would be very interesting to relate them to physical interaction that are not explicitly modeled in the considered simplified dynamics. In the next
Figure 12: First (above) and last (below) estimated horizontal wind fields of the sequence superimposed on observed pressure differences.
section we will show how this technique can be used to characterize low order dynamical system from noisy experimental data.

4 Characterization of turbulent flows

In this section, we describe how a data assimilation technique can be used to recover the parameters and the initial condition of a POD-Galerkin dynamical systems. Such systems, as previously explained are obtained considering a POD representation of the velocity fields in term of a mean flow and a linear temporal combination of spatial eigenfunctions:

\[ u(x, t) = \bar{u} + \sum_k a_k(t) \phi_k(x). \]

The low order system is obtained from a Galerkin projection on the subspaces associated to a low number of POD modes. The resulting projection leads in the standard case to a system of quadratic ordinary equation on the temporal coefficients (see section 1):

\[
\begin{aligned}
\frac{da_k}{dt} &= I_k + \sum_{i=1}^{M} L_{ik} a_i + \sum_{i=1}^{M} \sum_{j=1}^{M} C_{ijk} a_i a_j, \quad \forall k \in [1 \cdots M], \\
\phi_k(0) &= \langle u(x, 0) - \bar{u}(x), \phi_k(x) \rangle.
\end{aligned}
\]

The initial condition and the coefficient \(I\), \(L\), and \(C\) depends explicitly on on the POD modes and on the experimental data. In the standard approach, these coefficients are computed directly from the data and the POD modes. The system of ode’s is then integrated from the initial condition using an accurate Runge Kutta integration scheme. An improvement of this techniques consists to consider a least squares estimation scheme to estimate the coefficients of the dynamical system from the estimated temporal modes [4]. This technique works very well on DNS data but is much more prone to errors for experimental measurements.

An example of the results recovered by such a technique is presented on figure 4. This figure shows on the left illustration the trajectories of the two principal modes of an experimental wake behind a cylinder at Reynolds 150 (vortex shedding regime). The velocity measurements have been done through a PIV software.

On the right, reconstructed trajectory of the four first energetic modes are shown. As can be observed, the reconstructed trajectory are good only over a small period. Very quickly amplitude problems and dephasing arise. For very noisy observation associated to weaker energetic modes this dynamical system is even completely unstable and explodes. Let us note in addition that these simulations rely on an initial conditions that are obtained from the data 53 and may thus
The estimation of the low order dynamical system coefficients together with a denoising of the initial condition may be obtained relying on a variational assimilation process. The corresponding assimilation is based on the POD-Galerkin dynamical model and on the temporal modal coefficient extracted from the PIV data. The problem is formulated as an optimal control problem on the initial condition and the low order system coefficients. We seek for the value of the control variable that minimize the following cost function:

$$ J(I, L, C, \mathbf{a}(t_1)) = \sum_{j=t_1}^{t_{M}} \sum_{k=1}^{s} \left\| a_k(I, L, C, \mathbf{a}(t_1), t_j) - \hat{a}^{obs}_k(t_j) \right\|^2_{R(t_j)} $$  \hspace{1cm} (54)

The figure 4 shows on the left side the four first modes trajectories obtained with a damped system involving an additional artificial viscosity term. The right graphic presents the trajectories recovered after an assimilation process. For this assimilation the damped solution has been considered as an initial guess of the control process.

The phase portraits of the two and three first modes of the wake behind a cylinder are plotted on figure 15.

A demonstration of the system stability on a long time interval can be obtained using only a part of the data available. In figure 16, we show the trajectory recovered for 500 PIV motion fields over the 1000 fields available. The dynamical system has been then simulated along the whole time range (i.e. corresponding to the 1000 images).
Figure 14: Left: simulation of the four first modes after a least squares estimation of a damped dynamic system coefficients; Right: trajectories recovered for the four first modes after an optimal control process on the initial condition and on the dynamical system parameters.

Figure 15: Phase portraits of two (first row) and three (second row) temporal modes trajectories of a POD representation of a wake behind a cylinder at Reynolds 150. column (a): noisy coefficients estimated from PIV data; column (b): damped trajectories obtained after a least squares estimation of the dynamical system’s coefficients; column (c): trajectory obtained through the assimilation process.
Figure 16: Trajectories of the four first modes assimilated over the first half of the available PIV motion fields (500 frames) and simulated along the complete time range (1000 frames).

The modes trajectories are then compared to the rest of the unused data. The trajectories superimposed on the observation are presented on figure shows 17 Although a further accuracy could have been recovered integrating the complete set of data, this experiment demonstrates the stability of the results obtained. Otherwise, even if there is still at some places a significant deviation between the weaker energetic modes trajectories and the observed data, the recovered trajectories nevertheless allows us to get a coherent restitution of the experimental flows over a long time interval. The figure 4 compares the vorticity maps of PIV measurement restored through a gappy pod technique and vorticity maps obtained with a POD assimilation technique. In this example 10 modes have been used to represent the flow. As can be observed the vorticity maps are less noisy and more coherent in time.

A preliminary version this work has been published in [18]. Interested readers may find in it complementary results and a detailed description of the method principles.

5 Conclusion

In this workpackage we explored mainly two different methodological frameworks for the tracking or the assimilation of fluid flows from image sequences. The first one is constituted by
stochastic filtering approaches whereas the second one consists in variational assimilation techniques. Both kinds of methods have their own advantages and drawbacks. The stochastic techniques allows to get an estimation of an empirical distribution of the state variable given the data and as a consequence to provide an error covariance estimate. The technique is in addition recursive and can practically be used for control purpose if the associated computational burden is not to high. It is thus restricted to low dimensional state spaces and is, as a result, not well suited for analyses purposes of small motion scales. Ensemble Kalman filter shown a greater potential with respect to the restitution of small scales. However they are limited to linear Gaussian measurement. Variational assimilation allows to recover more accurate results. As they are inherently batch techniques they can only be used for off-line analyses. This last frameworks has allowed us to used simplified fluid dynamics complemented by additional control variable estimated from the image sequence. This process defines a kind of learning process for unknown terms of the dynamics. This could open a very interesting area of research in which one could try to specify for instance subgrid model from image data in Large eddies simulation. This technique has been also used to specify from PIV data a reduced dynamically system. Once again, data assimilation offered us a mean to learn for a given flow unknown terms of the dynamical system.

Figure 17: Trajectories of the four first modes assimilated over the first half of the available PIV motion fields (500 frames) and simulated along the complete time range (1000 frames).
Figure 18: Comparison of vorticity map reconstructions. Left column: vorticity obtained with a gappy POD restoration techniques of PIV measurements. Right column: vorticity maps obtained by the POD assimilation technique.
References


